Social Security deposits

Suppose you want to retire and make $p\%$ of your current annual income for the rest of your life. Suppose you expect that you'll live for L years after you retire. Suppose your annual salary while you're working is $\$N$. What fraction f of your salary should you put away every year, assuming you make an interest rate of $r\%$ annually? Assume you have to retire when you reach R years old, and that you start depositing money at year S.

Under these assumptions, how much money do you have to have when you retire, and how much will you have to set aside per year?

A formula for compound interest would help us here. In general, if you add $\$k$ to your bank account every year at $r\%$ interest, and you start with a principal of P, the amount of money P_n you'll have in year n satisfies this recurrence relation:

$$\begin{aligned} P_0 &= P \\ P_n &= (1+r)P_{n-1} + k, n > 0, \end{aligned}$$

which expands to

$$\begin{aligned} P_1 &= (1+r)P + k \\ P_2 &= P(1+r)^2 + k(1+r) + k \\ P_3 &= P(1+r)^3 + k(1+r)^2 + k(1+r) + k \\ &\vdots \end{aligned}$$

In general, assume

$$P_n = P(1+r)^n + kS_{n-1}, \quad n \geq 0, \quad S_n = \sum_{j=0}^{n} (1+r)^j$$

This is obviously true for $n = 0$ and $n = 1$. Assume it's true for all $n \geq 0$. Now $P_{n+1} = (1+r)P_n + k$ by the recurrence relation, whence

$$\begin{aligned} P_{n+1} &= (1+r)\left[P(1+r)^n + kS_{n-1}\right] + k \\ &= P(1+r)^{n+1} + k(1+r)S_{n-1} + k \end{aligned}$$

which agrees with the inductive hypothesis iff

$$S_n = (1 + r)S_{n-1} + 1$$

$S_n = S_{n-1} + (1 + r)^n$, so we have to determine whether

$$S_{n-1} + (1 + r)^n = (1 + r)S_{n-1} + 1.$$

First, note that

$$\begin{aligned} S_{n-1} &= \frac{1 - (1 + r)^n}{1 - (1 + r)} \\ &= \frac{(1 + r)^n - 1}{r} \end{aligned}$$

by a well-known identity. So the inductive hypothesis is true iff

$$\frac{(1 + r)^n - 1}{r} + (1 + r)^n = (1 + r)\frac{(1 + r)^n - 1}{r} + 1$$

A little laborious algebra confirms that the latter is true. So we've confirmed that

$$\begin{aligned} P_n &= P(1 + r)^n + kS_{n-1} \\ &= P(1 + r)^n + (k/r)[(1 + r)^n - 1] \end{aligned} \tag{1}$$

Quick sanity check: with $k = 0$, the compound-interest formula should equal $P(1 + r)^n$. It clearly does.

So back to the retirement problem. We want to guarantee that at retirement age R, you have as much money as you want. Once we know how much money we'll need to save, we can use the formula above to figure out how much money to add in each of the $(R - S)$ years that you're saving money.

How much money do we want to save? Assume we'll be making no income in the final L years of our life. We start with M dollars, say, and we draw down pN dollars every year, leaving M_n dollars in year n after we start drawing down. We want $M_L \geq 0$. So we need to find a formula for M_n.

Fortunately this just reduces to the compound-interest formula, with $k = -pN$, so we know

$$M_L = M(1 + r)^L - (pN/r)[(1 + r)^L - 1]$$

In order to guarantee $M_L \geq 0$, how large must M be?

$$\begin{aligned} M(1+r)^L &\geq (pN/r)[(1+r)^L - 1] \\ M &\geq \frac{pN[(1+r)^L - 1]}{r(1+r)^L} \end{aligned} \tag{2}$$

So that is our target amount of income at retirement; call it $\hat{M}$. Now how much must we put away each year in order to achieve $\hat{M}$? We put away fN dollars annually, with $0 < f < 1$, at an interest rate r, for a total of $R - S$ years. Again we use the compound-interest formula (1), with $P = 0$, $k = fN$, and $n = R - S$:

$$P_{R-S} = (fN/r)[(1+r)^{R-S} - 1] \geq \hat{M}$$

By (2), we need

$$\frac{fN[(1+r)^{R-S} - 1]}{r} \geq \frac{pN[(1+r)^L - 1]}{r(1+r)^L}$$

So how large must the fraction of our income be? That's easy enough:

$$f \geq \frac{p[(1+r)^L - 1]}{(1+r)^L[(1+r)^{R-S} - 1]}$$

Assume $L = 30$ (we'll live 20 years after retirement), $p = 0.75$ (we want to make 75% of our pre-retirement income), $r = 0.07$ (we expect to make 7% real interest every year), and $R - S = 40$ (we're saving money for 40 years). Then $f = 0.062$ – you have to set aside 6.2% of your money to achieve your goal.