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EXCHANGE PROCESSES WITH PRICE ADJUSTMENT

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Section 1

The goal is to show:

Main result. *In a pure exchange economy, an exchange, price adjustment process, responsive to transaction costs, which does not stop unless forced to by market conditions, converges to a price equilibrium. There exist such processes starting from any state of any (pure exchange) economy.*

Before we present the mathematical model (section 2), we give some discussion. Studied here is a non-tatônnement process in a pure exchange economy with price systems. Our treatment has much of the spirit of works published in 1962 by Uzawa, Hahn, Hahn and Negishi, Morishima and surveyed in Arrow–Hahn (1971, ch. 13). See especially the last reference for the history and discussion of the economic side of this problem.

It is also the case that our treatment has a number of new and different features from the 1962 work. For one thing, our economic process is not supposed to satisfy some ordinary differential equation. Such a hypothesis implies too deterministic a setting for a state of an economy moving in time. Economics seems to be different from theoretical physics where the state at one time determines the state at all future times. Economic forces don't seem to have such a Newtonian flavor and one has to be careful about carrying over methods from physics to economics.

Another consideration is that there is no (apparent, at least) economic principle which gives us a differential equation. At best, an assumption that an economic process satisfies a differential equation should be treated similarly to the assumption that preference relations can be represented by utility functions.

Rather than differential equations, our work focuses on cone fields given essentially by non-linear differential inequalities and equalities.In this way we have been able to minimize or eliminate ad hoc hypotheses.

The last sentence of our result shows that the processes we work with exist in a great variety of situations. This kind of theorem seems not to have been shown in the earlier models.

What can one say about an economic interpretation of this model? My closest experience with such a market is a "mineral bourse" or large mineral show, and the essence of the economic process of this market seems to fit into the work of this paper. This kind of market lasts two or three days; initially, buyers, dealers, traders (many agents are all three) bring mineral specimens of some value and/or money. When the show starts, minerals are traded, bought and sold, more quickly at first, and one sees price equilibrium reached in the afternoon of the last day as prices stabilize and exchange slows to a halt. The exact equilibrium depends on factors such as which agents first encounter each other.

Of course, money also plays a role in the mineral bourse, but I believe our model here could be developed to include money on that level. Also our model does not deal with the detailed mechanisms of exchange and price adjustment. Again this can probably be dealt with in our framework. Both of these seem interesting lines of investigation. The work of F. Fischer (1972] is relevant.

We would see some kind of behavioral strategy as the force behind agents' moves in explicating the model here.

The kind of goods that don't fit into this model are those like labor, or perishable ones. Our feeling is that the theory of economic processes is best developed in two parts depending on two idealizations of goods into durable and perishable classes. This paper deals with the durable class and Smale (forthcoming) is concerned more with a way of looking at the perishable class. A theory of production processes may have to do with bringing the two together.

To elaborate a little on the above "main result", a *process* is given by a path over time in the space of states of a pure exchange economy. The path associates to each time an allocation of resources of the economy and prices for each good.

This process, in order to qualify as an exchange price adjustment process, is supposed to satisfy two hypotheses, an exchange axiom and a price-adjustment axiom. The *exchange axiom* asserts that (a) the total resources of the economy are constant (there is no production), (b) exchange takes place at current prices, (c) an exchange increases satisfaction of the participating agents, and (d) some exchange will take place, provided it is possible consistent with (a), (b) and (c).

The price adjustment axiom is defined in terms of a short-run version of demand. A total excess demand approach requires long-run optimization on the part of the agents and is less consistent with our spirit, which is closer to that of behavioral strategies. At given prices and goods processed, one defines the short-run demand of an agent to be the direction his preferences take him when restricted to his budget set.

The *price-adjustment axiom* asserts that prices adjust in the direction of some weighting of the short-run demands of all the agents.

Classical hypotheses of convexity and monotonicity on the preferences of agents are assumed, as well as differentiability. Also, we require a boundary condition on the preference relation which prevents an agent from trading off every last bit of some commodity.

A Walrasian price equilibrium in a pure exchange economy depends on the trader's endowments. Thus if one allows a real passage of time, say, an actual exchange to take place, and several such, this initial endowment becomes lost in the shuffle. Thus if one allows this kind of time passage ("non-tatônnement") one must replace a Walrasian price equilibrium by a different notion of price equilibrium. For our purposes, i.e., as stated in our main result above, a *price equilibrium* is a feasible allocation and price system where, for each agent, satisfaction is maximized on his budget set defined relative to his wealth at equilibrium. Equivalently, a price equilibrium is an optimal allocation together with a supporting price system.

I first spoke on this non-tatônnement process four years ago and during this passage of time, it has developed into the form presented here. Discussions with many economists have been very helpful in this development. The first ideas (with cone fields) are in Smale (1974), and a later form is in Smale (forthcoming).

Section 2

Here we give the mathematical formulation of the results expressed in section 1.

A description of the economy with our assumptions goes as follows. Take *commodity space* P as

$$P = \{x \in R^l | x = (x^1, \ldots, x^l), \quad x^i > 0\}.$$

We suppose that preferences for the ith consumer can be represented by a C^2 utility function $u_i \colon P \to R$, $i = 1, \ldots, m$, with the following properties:

Monotonicity (differentiable version). The gradient of u_i at x, $\operatorname{grad} u_i(x)$, is in P, for each x in P.

Convexity (differentiable version). The second derivative $D^2u_i(x)$ (as a bilinear symmetric form) on the space

$$\{v \in R^l | v \cdot \operatorname{grad} u_i(x) = 0\}$$

is negative definite. We are using the inner product on R^l in the expression $v \cdot \operatorname{grad} u_i(x)$.

Boundary condition. For each $c \in R$, $u_i^{-1}(c)$ is a closed subset of R^l.

These conditions have been much discussed in some recent literature [see Debreu (1972) or Smale (1976a)]. Possibilities for their relaxation will be discussed (section 5).

While the three assumptions are expressed in terms of a utility function, these properties depend only on the underlying preference relation. Let

$$S_+ = \{x \in P \mid \|x\| = 1\},$$

where $\|x\|^2 = \sum (x^i)^2$. If $u_i\colon P \to R$ is a utility function satisfying the above conditions, define $g_i\colon P \to S_+$ by $g_i(x) = \operatorname{grad} u_i(x)/\|\operatorname{grad} u_i(x)\|$. Then g_i can be defined by the indifference surfaces of u_i directly as the unit normal, and we will use the g_i which have a utility representation as above in the following.

The economies we consider will have fixed total resources denoted by $r \in P$. An *economy* means the data $(r, g_1, \ldots, g_m)$ as described.

The space of (feasible) *allocations* is then

$$W_r = W = \{x \in (P)^m \mid x = (x_1, \ldots, x_m), \quad x_i \in P, \quad \textstyle\sum x_i = r\},$$

using vector notation. (In section 3 where prices are ignored, W is the set of states of an economy.) The Pareto optimal points in W form a submanifold of dimension $m-1$ which is denoted by θ [Smale (1976a)].

Prevailing prices will be elements of S_+ and so we speak of *states* of our economy as elements (x, p) of $W \times S_+$, where $x = (x_1, \ldots, x_m)$ is an allocation and $p = (p^1, \ldots, p^l)$ is a price system. Here x_i denotes the vector of goods of the ith agent and p_j is the price of one unit of the jth good.

An economic *process* in this context is a C^1 map ϕ: $[a, b) \to W \times S_+$, where $\phi(t) = (x(t), p(t))$ is the state of the economy at time t. Such a process can be thought of as a continuous sequence of exchanges and price adjustments. We have chosen $b < \infty$ and left the interval open ended. If one used a closed interval $[a, b]$, we would be in a somewhat similar situation except that convergence as $t \to b$ would be a foregone conclusion. We prefer not to have such a built-in hypothesis.

The exchange axiom on the process, $[a, b) \to W \times S_+$, $t \to (x(t), p(t))$, takes the following mathematical form:

Exchange axiom

(a) $p(t) \cdot x_i'(t) = 0$, each $t \in [a, b)$ and $i = 1, \ldots, m$ (or exchange takes place at prevailing prices). Here x_i' or $x_i'(t)$ is $\mathrm{d}x_i/\mathrm{d}t\,(t)$ and we are using the dot product.

(b) If $x_i' \neq 0$, then $x_i' \cdot g_i(x_i) > 0$, for each t and i [if agent number i exchanges some goods, he increases satisfaction; note by the chain rule, $x_i' g_i(x_i) > 0$ if and only if $\mathrm{d}/\mathrm{d}t\, u_i(x_i(t)) > 0$].

(c) Some $x_i' \neq 0$ if possible, consistent with (a) and (b) (some agent will trade if the market situation permits).

To spell out (c), at t, some x_i' is not zero if there exists a non-trivial solution in $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$, $\bar{x}_i \in R^l$, of the following system of equations:

$$\begin{aligned}
&\sum \bar{x}_i = 0, \\
&p(t) \cdot \bar{x}_i = 0, \qquad i = 1, \ldots, m, \\
&\bar{x}_j \cdot g_j(x_i(t)) > 0, \qquad \text{if} \quad \bar{x}_j \neq 0, \quad \text{each } j.
\end{aligned}$$

Implicit already in our notion of process is the condition that the total resources are constant or that $\sum x_i(t) = r$.

For a process to qualify as an *exchange/price-adjustment* process in addition to the previous exchange axiom, it must satisfy a price adjustment axiom. Toward this end, define the *short-run demand* $d_i(x_i, p)$ of agent i with goods $x_i \in P$ at prices $p \in S_+$ by

$$d_i(x_i, p) = \pi_p g_i(x_i),$$

when $\pi_p \colon R^l \to R^l$ is the orthogonal projection of R^l along p onto the linear subspace $p^\perp$ of vectors orthogonal to p. We are only concerned with the direction of $d_i(x_i, p)$, not the magnitude. Another way of expressing $d_i(x_i, p)$ is to consider the budget set of agent i with goods x_i at prices p,

$$B = \{y \in R^l \mid p \cdot y = p \cdot x_i\}.$$

Restrict the utility function u_i to B and take the gradient at x_i, $\operatorname{grad} u_i|_B(x_i)$. Then there is $\lambda > 0$ so that

$$\operatorname{grad} u_i|_B(x_i) = \lambda d_i(x_i, p).$$

Thus d_i is the direction of increasing satisfaction fastest at given prices. The price adjustment axiom says that prices change according to forces generated by these short-run demands. Precisely, say that a process $t \to (x(t), p(t))$ satisfies the price adjustment condition if:

Price-adjustment axiom

There is some $\varepsilon > 0$, and for each $t \in [a, b)$, there is a set of numbers $l_i > \varepsilon$, $l_i < 1/\varepsilon$, $i = 1, \ldots, m$, with $p'(t) = \sum l_i d_i(x_i, p)$.

Here the ε imposes a slight uniformity. Roughly speaking, if these demand forces are in the same general direction, the prices adjust correspondingly, while if the infinitesimal demands are in opposing directions, no constraint is imposed on $p'(t)$.

It is easy to construct an example of an oscillating curve where $x(t)$ doesn't converge to anything. However, these examples can't happen near the Pareto optimal set or if a hypothesis is made reflecting some idea of transactions costs. We make this precise by giving mathematical content to the condition on the process "responsive to transaction costs". First the condition is given in terms of utility functions. The process $[a, b) \to W \times S_+$, $t \to (x(t), p(t))$ is *responsive to transaction costs if* either

(a) $\lim t \to b\ x(t) \cap \theta \neq \phi$, where θ is the set of Pareto optimal points, and $\lim_{t \to b} x(t)$is the set of all limit points $x(t_i)$ for all sequences $t_i \to b$;

or

(b) there is $\delta > 0$, so that $\sum [u_i(x_i(t_1)) - u_i(x_i(t_0))] \geqq \delta \sum \|x_i(t_1) - x_i(t_0)\|$ for all pairs $t_0, t_1 \in [a, b)$ with $t_0 < t_1$.

This says that (away from θ), relative to the size of the transaction, there is a total gain in satisfaction, perhaps small.

An equivalent way of expressing this condition is, off of some neighborhood of θ, the sum of the angles between $x_i'(t)$ and the indifference surfaces through $x_i(t)$ are bounded uniformly away from zero as $t \to b$.

Note that we have not defined the notion of transaction costs itself and that this condition for the process depends only on the preference relations and not their particular representation by utility functions.

We add that if the process is assumed to satisfy an ordinary differential equation on $W \times S_+$, this condition could be eliminated.

Finally, note that this condition does not involve prices, just the exchange process.

What should the phrase mean, "process doesn't stop unless forced to by market conditions"? How should this condition be modeled?

Suppose, for example, that an exchange/price-adjustment process $(x(t), p(t))$ converges to (x_0, p_0) as t tends to b and yet a non-trivial exchange/price-adjustment process can start at (x_0, p_0). Then surely the process has stopped prematurely or is incomplete.

But there is an easy example of an exchange/price-adjustment process $(x(t), p(t))$ such that $x(t) \to x_0$ while $p(t)$ doesn't converge and yet a non-trivial exchange process can start from x_0, p_0 for any p_0 in the limit set of $p(t)$. Thus in this case exchange is possible for any limiting price system and so the process should be considered incomplete. These considerations motivate the following definition:

An exchange/price-adjustment process $[a, b) \to W \times S_+$, $t \to (x(t), p(t))$, is said to be *incomplete* if

(a) $(x(t), p(t))$ converges to (x_0, p_0), and a non-trivial exchange price adjustment process starts from (x_0, p_0);

or

(b) $x(t)$ converges to x_0, and for any limiting price system $p_0 \in \lim p(t)$, non-trivial exchange can take place at (x_0, p_0) in accordance with the exchange axiom.

A process which is not incomplete will be called *complete.*

Thus "complete" formalizes the condition "stops only when forced to by market conditions".

Recall that a (non-tatônnement) price equilibrium is a pair $(x, p) \in W \times S_+$ which satisfies the conditions $g_i(x_i) = p$ for $i = 1$ up to m. The set of price equilibria forms an $(m-1)$-dimensional submanifold Λ of $W \times S_+$ [Smale (1976a)].

Now we can give a mathematically precise restatement of the main theorem of section 1.

Theorem. An exchange price adjustment process $[a, b) \to W \times S_+$, $t \to (x(t)),$ $p(t))$, *complete and responsive to transaction costs, must converge to a price equilibrium as t tends to b. There exists such processes starting from any state of any (pure exchange) economy.*

The proof will be given in section 4. Meanwhile, a simpler version of this result (no prices) will be stated and proved (section 3).

Section 3

We consider here the main theory of the paper for the case of a pure exchange economy, but no price systems. The situation is somewhat simpler than with prices and yet many of the basic ideas are present. In the next section, prices will be added. The notation and the hypotheses on the economy set down in section 2 will be used. Here complete proofs of the results will be given.

Thus W will denote the set of states (for the purposes of this section) of a pure exchange economy with fixed total resources $r \in P$. An economic process is given by a map

$$[a, b) \to W, \ t \to x(t) = (x_1(t), \ldots, x_m(t)).$$

This process satisfies the *exchange axiom* (compare with the previous section), provided:

Exchange axiom. For each t in $[a, b)$,

(a) $x_i' \cdot g_i(x_i) > 0$, if $x_i' \neq 0$, $i = 1, \ldots, m$.
(b) Some $x_i' \neq 0$, if there exists a non-trivial solution in $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$, $\bar{x}_i \in R^l$ of the following system:

$$\sum \bar{x}_i = 0$$

$$\bar{x}_i \cdot g_i(x_i) > 0, \qquad \text{if} \quad \bar{x}_i \neq 0, \quad i = 1, \ldots, m.$$

This means (a) for someone to trade, he must improve his lot, and (b) if trade is feasible consistent with (a), some trade will take place.

Built into the construction is the condition that $\sum x_i = r$.

A curve $t \to x(t)$ satisfying the exchange axiom will be called an *exchange curve*. An exchange curve $t \to x(t)$ is called *incomplete* if $\lim_{t \to b} x(t) = x_0$ and a non-trivial exchange curve starts at x_0. An exchange curve is called *complete* if it is not incomplete. *Responsive to transaction costs* is the same as in section 2.

Theorem. Suppose $[a, b) \to W$, $t \to x(t)$ is an exchange curve responsive to transaction costs and complete. Then as t tends to b, $x(t)$ converges to a Pareto optimal point of W. Furthermore, given any economy, as in section 2, and any non-optimal state, there exists a complete exchange curve, responsive to transaction costs, starting at that state.

Toward the proof of the theorem, we elucidate the exchange axiom. For each $x \in W$, a cone C_x^* of vectors will be defined (it can be thought of as a set of tangent vectors to W at x). This cone depends on the economy and x but not a process (i.e., an exchange curve). Let

$$C_x^* = \{\bar{x} \in (R^l)^m | \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m),$$
$$\bar{x}_i \in R^l, \quad \sum \bar{x}_i = 0, \quad \bar{x}_i \cdot g_i(x_i) > 0 \quad \text{or} \quad \bar{x}_i = 0, \quad \text{each } i\}.$$

It can be seen from the definitions that:

Fact. A curve x: $[a, b) \to W$ is an exchange curve if and only if $x'(t) \in C_x^*(t)$ each t and $x'(t) \neq 0$ if $C_{x(t)}^* \neq 0$.

This (cone) C_x^* plays a key role in this paper. It is neither open nor closed.

Proposition 1

(a) *C_x^* is a convex cone.*
(b) *$C_x^* = 0$ if and only if x is Pareto optimal.*

(c) *The function* $\sum u_i$ *is a strictly increasing function along an exchange curve at all non-optimal points.*

(d) *If* $t \to x(t)$ *is an exchange curve and* $x(t_0)$ *is optimal, then* $x(t) = x(t_0)$ *for all* $t \geqq t_0$.

The proof of (a) is routine and simple. For (b), one uses the characterization of θ, the set of Pareto optimal points by the first-order condition [see, e.g., Smale (1976a)] : $x \in \theta$ if and only if all the $g_i(x_i)$ coincide. Let

$$C_x = \{\bar{x} \in (R^l)_m | \sum \bar{x}_i = 0; \quad \bar{x}_i \cdot g(x_i) > 0, \quad \text{each } i\}.$$

Then it is easily seen and known that x is Pareto optimal if and only if C_x is empty. Since $C_x^* \supset C_x$, if $C_x^* = 0$, then C_x is empty and so x is optimal. On the other hand, let x be optimal; then all the normalized gradients $g_i(x_i)$ coincide. So if $\bar{x} \in C_x^*$, and $\bar{x} = (\bar{x}, \ldots, \bar{x}_m)$, then all the non-zero $\bar{x}_i$ lie in the same open half space; and $\sum \bar{x}_i = 0$ implies in fact that each $\bar{x}_i$ is zero. This proves (b).

For part (c), let $t \to x(t)$ be an exchange curve. Then the condition that $x_i'(t) \cdot g_i(x_i(t)) \geqq 0$ all t, implies that u_i is non-decreasing along the curve, each i. On the other hand, if $x(t) \notin \theta$, $x_i' \cdot g_i(x_i) > 0$ for some i; thus at that t, u_i and $\sum u_i$ are strictly increasing. This proves (c). Then (d) follows. The proposition is proved.

Proposition 2. Suppose the exchange curve $x \colon [a, b) \to W$ *is responsive to transaction costs. Then* $\lim_{t \to b} x(t)$ *exists.*

Proof. Consider first the case that the curve $t \to x(t)$ stays outside some neighborhood of θ, and let α, β belong to the limit set $\{\lim x(t) | t \to b\}$. Suppose $\alpha \neq \beta$, and let $d = \|\alpha - \beta\|$. There is an increasing sequence t_j such that $\|x(t_{2i}) - \alpha\| < d/4$ and $\|x(t_{2i+1}) - \beta\| < d/4$ for all i. Thus $\|x(t_{2i+1}) - x(t_{2i})\| > d/2$ for all i. From the transaction cost condition there is some k with $u_k(x_k(t)) \to \infty$. From the boundary condition and total resource constraint this is impossible. Thus $\alpha = \beta$, and in this case $\lim_{t \to b} x(t)$ exists.

Next consider the case that $\{\lim_{t \to b} x(t)\} \cap \theta \neq \phi$, and let $\alpha \in \theta$ satisfy $\alpha \in \lim x(t)$, with $\alpha = (\alpha_1, \ldots, \alpha_m)$. Suppose $\beta \neq \alpha$ is also in this limit set. Let $t_k \to b$ be an increasing sequence such that $x(t_{2j}) \to \alpha$, $x(t_{2j+1}) \to \beta$. For any i, since $u_i(x_i(t_{2j})) \to u_i(\alpha_i)$ and $u_i(x_i(t))$ is non-decreasing, it follows that $u_i(\alpha_i) = u_i(\beta_i)$. Since α is optimal, then β also is; $\beta \in \theta$. But u on θ is one to one [see, e.g., Smale (1976a)]. Therefore $\alpha = \beta$, proving Proposition 2.

Proposition 3. (Extension Proposition). Given any $x_0 \in W$, $\alpha_0 \in C_{x0}^*$, $\alpha_0 \neq 0$ *if* $C_{x0} \neq 0$, *there is a function* α *which assigns to each* $x \in W$, $\alpha(x) \in C_x^*$, $\alpha(x) \neq 0$

if $C_x^* \neq 0$ *and* $\alpha(x_0) = \alpha_0$. *Furthermore,* $\|\alpha(x)\| \leqq \alpha_0\|+1$ *for all* $x \in W$, α *is continuous and* C^1 *off of* θ. *In other words,* α *is a* vector field *on* W, *extending* α_0 *with the properties mentioned.*

Proof. By an easy version of the Whitney extension theorem [see Golubitsky–Guillemin (1973, p. 17)] there is a C^1 function ρ on W, with values in [0, 1], which is zero exactly on θ, and is 1 at x_0 if $x_0 \notin \theta$.

If β can be defined on the set $W-\theta$ satisfying the properties of Proposition 3, then we are finished by defining

$$\alpha(x) = \rho(x)\,\beta(x), \qquad \text{for} \quad x \in W-\theta,$$

$$\alpha(x) = 0, \qquad \text{for} \quad x \in \theta.$$

Now β can be constructed locally as in the proposition by a constant extension and a partition of unity using a locally finite covering [as in Golubitsky–Guillemin (1973) for example] yields a globally defined β since the cones C_x^* are convex. Just note that in this process, if $\alpha(x) \in C_x^*$, $\alpha(x) \neq 0$, then $\alpha(x) \in C_y^*$ for y near x.

We are now ready to prove the theorem of this section.

Let be given $t \to x(t)$ as in the first part of that theorem. Then by Proposition 2, $x(t)$ converges as $t \to b$ to a point, say x_0, in W. Suppose that $x_0 \neq \theta$. Then $C_{x_0}^* \neq 0$ by Proposition 1. Let non-zero $\alpha_0 \in C_{x_0}^*$ and apply Proposition 3 to obtain an extension α to W. Solve the ordinary differential equation $x' = \alpha(x)$ to obtain a solution $t \to x(t)$ with initial condition $x(0) = x_0$. Since $x'(t) = \alpha(x(t)) \in C_{x(t)}^*$ $x(t)$ is an exchange curve (non-trivial) starting from x_0. But this couldn't happen since our exchange curve was assumed complete to begin with. Thus $x_0 \in \theta$, and the first part of the theorem is proved.

For the second part of the theorem, given a state $x_0 \notin \theta$ of some economy, we take any non-zero α_0 in $C_{x_0}^*$. By Proposition 3, extend α_0 to α as in the preceding paragraph. Solve the ordinary differential equation $x' = \alpha(x)$ with initial condition $x(0) = x_0$. This yields an exchange curve $t \to x(t)$ with $x(0) = x_0$. Along any non-trivial solution curve $\sum u_i$ is a strictly increasing function by Proposition 1. Also $\lim x(t)$ is not empty by the boundary condition and total resource limitation [$x(t)$ stays in a compact set]. Under these conditions, it follows from properties of differential equations that if $y \in \lim_{t \to -} x(t)$, then $\alpha(y) = 0$. Thus $y \in \theta$, and as in Proposition 2, $x(t)$ converges to y. By a reparameterization as in Schechter (1975), x can be assumed defined on the interval [0, 1), thus finishing the proof of the theorem.

See also Schechter (1975) for related theorems.

Section 4

We give the proof of the theorem of section 2 (and thus the main result of section 1).

The first part of this proof depends on the construction of a field of cones over $W \times S_+$ corresponding to the exchange and price adjustment axioms. More precisely, for each (x, p) in $W \times S_+$ define

$$B_{x,p} = \{\bar{x} \in C_x^* | \bar{x}_i \cdot p = 0, \text{ each } i\}.$$

Let $A_{x,p} = B_{x,p} = 0$ if $B_{x,p} \neq 0$. Otherwise let $A_{x,p} = B_{x,p} = 0$. Here C_x^* has been defined in the previous section. Note that $t \to (x(t), p(t))$ satisfies the exchange axiom of section 2 if and only if $x'(t) \in A_{x(t),\, p(t)}$ for all t.

Define for each $\varepsilon > 0$,

$$D_{x,p}(\varepsilon) = \left\{\bar{p} \in R^l | \bar{p} \cdot p = 0, \quad \bar{p} = \sum_{i=1}^{m} l_i d_i(x_i, p), \quad \varepsilon < l_i < \frac{1}{\varepsilon}\right\},$$

and

$$D_{x,p} = \bigcup_{\varepsilon > 0} D_{x,p}(\varepsilon).$$

Then $t \to (x(t), p(t))$ satisfies the price adjustment axiom of section 2 exactly if there is some $\varepsilon > 0$ and $p'(t) \in D_{x(t),\, p(t)}(\varepsilon)$ for all t.

Write $\varphi = (x, p)$ and let

$$Y_\varphi(\varepsilon) = A_\varphi \times D_\varphi(\varepsilon) \supset (R^l)^m \times R^l$$

and

$$Y_\varphi = \bigcup_{\varepsilon > 0} Y_\varphi(\varepsilon).$$

Proposition 1

(a) *$Y_{x,p}$ is a convex cone.*
(b) *$Y_{x,p} = 0$ if and only if (x, p) is a price equilibrium.*
(c) *If $D_{x,p}$ contains 0, but isn't 0, then $A_{x,p} \neq 0$.*
(d) *$(x, p) \in A$ if and only if $D_{x,p} = 0$.*

Note especially (c) which asserts that if $p' = 0$ (prices are not changing) and (x, p) is not an equilibrium, then $A_{x,p} \neq 0$ so that exchange takes place in accordance with the exchange axiom.

For the proof, the first-order conditions [see, e.g., Smale (1976a)] assert that for $(x, p) \in W \times S_+$, $(x, p) \in A$ if and only if $g_i(x_i) = p$ for all i. But $D_{x,p} = 0$ if and only if all the $d_i(x_i, p) = 0$ or equivalently $g_i(x_i) = p$, all i. This gives (d).

Of course if $Y_{x,p} = 0$ then $D_{x,p} = 0$ and so $(x, p) \in \Lambda$. Thus for (b) it is sufficient to show that if $(x, p) \in \Lambda$ and $D_{x,p} = 0$ then $A_{x,p} = 0$. But if $(x, p) \in \Lambda$ then x is optimal [the fundamental theorem of welfare economics, e.g., Smale (1976a)] so $C_x^* = 0$ (compare section 3). This implies (b).

The proof of (a) is direct and we omit it.

It remains to prove (c). By the hypothesis of (c), we may write $\sum l_i d_i(x_i, p) = 0$ where $l_i d_i(x_i, p) \neq 0$ for some i. Define $\bar{x}_i = l_i d_i(x_i, p)$, each i, and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$. Then $\bar{x} \neq 0$ and $\sum \bar{x}_i = 0$. Since $d_i = \pi_p g_i(x_i)$, it follows that if $\bar{x}_i \neq 0$, then $\bar{x}_i \cdot g_i(x_i) > 0$. Therefore $\bar{x}$ is a non-zero element of $A_{x,p}$. We have constructed in $\bar{x}$ an infinitesimal trade. Proposition 1 is proved.

For $x \in W$ let D'_x be the cone in R^l generated by $g_i(x_i)$, $i = 1, \ldots, m$. Thus

$$D'_x = \{p \in R^l \mid p = \sum l_i g_i(x_i), l_i \geqq 0\}.$$

Proposition 2. Let $t \to (x(t), p(t))$ *be an exchange price adjustment process with* $x(t) \to x_0$ *as* $t \to b$. *Assume* $\lim_{t \to b} p(t)$ *doesn't exist. Then every* p_0 *in the limit set,* $\lim p(t)$ *satisfies*

(a) $p_0 \in D'_{x_0}$, *and*
(b) p_0 *is not a vertex of* D'_{x_0}.

One may restate (b) *as* (b′) *if* $p_0 = g_i((x_0)_i)$ *for some* i, *then* $g_i((x_0)_i) = \sum_j l_j g_j((x_0)_j)$ $l_j \geqq 0, g_j((x_0)_j) \neq g_i((x_0)_i)$.

We give this proof in outline with details to be filled in by an interested reader. First, let

$$d_x(p) = \inf_{q \in D'_x} \sum (p_i - q_i)^2,$$

where p_i, q_i are the coordinates of p and q, respectively.

It can be shown from the price adjustment axiom that if $d_x(p(t))$ is positive, then it decreases as t increases.

Next one shows that near p_0 in Proposition 2, p' must lie in a field of uniformly pointed cones. Putting these two pieces of information together yields a proof of Proposition 2.

The rough idea in the following proposition is to show if $p \in D'_x$ then $A_{x,p} \neq 0$, via the route that $p \in D'_x$ if and only if $0 \in D_{x,p}$ and then apply Proposition 1(c). However, these statements are not exactly correct. We proceed:

Proposition 3. If $p \in D'_x$ *and* p *is not a vertex of* D'_x, *then* $A_{x,p} \neq 0$.

Proof. We need to sharpen the argument of Proposition 1(c) slightly.

Let J be the set of j such that $p = g_j(x_j)$. If J is empty, then let $p = \sum l_i g_i(x_i)$, $l_i \geqq 0$, where not all the l_i are zero. Apply π_p to obtain $0 = \sum l_i d_i(x_i, p) = \sum \bar{x}_i$. Since some $\bar{x}_i$ is not 0, one can finish as in Proposition 1(c).

Suppose now J is not empty and let $j \in J$. Since p is not a vertex,

$$p = g_j(x_j) = \sum_{i \in I} k_i g_i(x_i), \qquad k_i > 0.$$

Here I is a non-empty indexing set with the property if $i \in I$, then $g_i(x_i) \neq g_j(x_j)$. Application of π_p yields

$$0 = \sum_{i \in I} k_i d_i(x_i, p) = \sum_{i \in I} \bar{x}_i,$$

with $\bar{x}_i$ not zero and again one finishes as in 1(c). This proves the proposition.

Proposition 4. (Extension Lemma). Let $(x_0, p_0) \in W \times S_+$ and $(X_0, \pi_0) \in Y_{x_0,p_0}$. Then there exists an extension (X, π) of (X_0, π_0), $(X(x, p), \pi(x, p)) \in Y_{x,p}$ with X, π continuous in x, p, π being C^1 and X being C^1 off of the set where $X = 0$. Thus $X(x_0, p_0) = X_0, \pi(x_0, p_0) = \pi_0$.

Proof. We construct first π, then X. Since $D_{x,p}$ is a cone, if π is constructed locally, a partition of unity will finish the construction. Thus let $(x^*, p^*) \in W \times S_+$. If $(x^*, p^*) = (x_0, p_0)$, let $\pi(x^*, p^*) = \pi(x_0, p_0)$. Otherwise let $\pi(x^*, p^*)$ be an arbitrarily chosen element of D_{x^*,p^*}. In either case we can write $\pi(x^*, p^*) = \sum l_i d_i(x_i^*, p^*)$ with $l_i > 0$. For (x, p) near (x^*, p^*), let $\pi(x, p) = \sum l_i d_i(x_i, p)$. This gives a desired local extension and hence a global extension for π.

Now let

$$Q = \{(x, p) \in W \times S_+ | A_{x,p} = 0\}.$$

We will show that Q is a closed set, or that the complement of Q is open. Let $(x^*, p^*) \notin Q$ and $(\bar{x}', \bar{p}') \in A_{x^*,p^*}$; let J be the (non-empty) set of i with $\bar{x}_i' \neq 0$.

Consider for x, p near (x^*, p^*) the following system of equations for $\bar{x}_i \in R^l$, $i = 1, \ldots, m$:

(i) $\bar{x}_i = 0, \qquad$ for $\; i \notin J$,

(ii) $\bar{x}_i \cdot p = 0, \qquad$ for each $\; i \in J$,

(iii) $\sum_{i \in J} \bar{x}_i = 0$,

(iv) $\bar{x}_i \cdot g_i(x_i) > 0, \qquad$ for $\; i \in J$.

By linear algebra, since (ii) and (iii) have a solution for $p = p^*$, a solution will exist for p near p^*. Furthermore, this solution will be close to the original x_i'. Therefore, (iv) will be satisfied for x near x^*.

This proves our assertion that Q is a closed set.

Next, construct $\rho: W \times S_+ \to [0, 1)$ which is zero exactly on Q, continuous and C^1 off of Q. The proof is finished just as in Proposition 3 of section 3.

Now to the proof of the main result. Let $t \to (x(t), p(t))$ be an exchange price adjustment process, complete and responsive to transaction costs. Then by Proposition 2 of section 3, we have convergence $x(t) \to x_0$. If $p(t)$ converges also, say to p_0, then $(x_0, p_0) \in A$. Otherwise, application of Proposition 4 in the same way as the extension lemma was used in section 3, would violate completeness

Suppose on the other hand $p(t)$ doesn't converge to a point. Let some price system p_0 be in the limit set. Apply Proposition 2 to see that $p_0 \in D'_{x_0}$, but that p_0 is not a vertex. Now apply Proposition 3 to see that A_{x_0,p_0} is not zero. Finally, application of Proposition 4 shows that completeness is violated. This finishes the proof.

Section 5

We indicate briefly and tentatively how a discrete version of the main result could go. This version allows exchange at different prices at the same time and has some other advantages; perhaps one could eventually dispense with differentiability using this approach.

Think of the real numbers R as literally time and let J be the set of positive integers; some bounded map $\alpha: J \to R_+$, such that $\alpha(q+1) \geqq \alpha(q)$, for each $q \in J$. A *process* is then some α as above and a map $J \to W \times S_+$ where $x(q) \to x(q+1)$ is interpreted to be an exchange at prices $p(q)$ at date $\alpha(q)$, for $q \in J$.

To qualify as an *exchange price adjustment process*, the process must satisfy the following two axioms.

Exchange axiom

(a) $p(q) \cdot (x_i(q+1) - x_i(q)) = 0$.

(b) $u_i(x_i(q+1)) \geqq u_i(x_i(q))$ with strict inequality if $x_i(q+1) \neq x_i(q)$.

(c) Given q, $x(q)$, $p(q)$, then $x(q+1) \neq x(q)$ provided there is some $x(q+1) \neq x(q)$ satisfying (a) and (b).

Price-adjustment axiom. There is some $\varepsilon > 0$ such that $p(q+1) - p(q) = \sum l_i d_i(x_i(q), p(q))$, $\varepsilon < l_i < 1/\varepsilon$, each i. The process will be called *responsive to transaction costs* provided either $(\lim x(q)) \cap \theta \neq \phi$ or there is a $\delta > 0$ such that $\sum (u_i(x_i(q+1)) - u_i(x_i(q)) \geqq \delta \|x(q+1) - x(q)\|$ for each q.

Completeness is defined analogously to the definition of section 2.

One also defines the notion of "localizable", or *behaviorally oriented* for the process to mean that given any $\delta > 0$, one can interpolate a new exchange/price-adjustment process by adding intermediate trades and adjustments so that each move is less than δ.

Then with the added hypothesis of localizable from the previous paragraph, the main theorem of section 2 would seem to be true in this context also.

Return now to consider the problem of improving on the main result of section 1. There exist numerous examples to show no hypothesis can be dropped and the conclusions remain valid. However, there are definite limitations in our hypotheses. Some have been discussed previously.

The boundary condition on the preference relation, even though it has been used frequently in mathematical economics, is hard to justify on economic grounds. On the other hand, there has been enough work in differentiable mathematical economics with mild boundary conditions, so that this limitation does not seem to be serious. Schechter's (1975) developments in this direction are the most complete to date.

If one were to abandon the convexity hypothesis of the preference relation, I believe with slight modifications the theory would go through. The convergence would lead, however, to an allocation which is locally optimal and to a state which would not be a price equilibrium, but only an extended price equilibrium in the sense of Smale (1974).

Probably one could also replace a price system out of equilibrium with some kind of cone of possible prices.

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